$$\Leftrightarrow 4p^2 + r^2 + 8Rr \geq 5(r^2 + 12Rr) \Leftrightarrow p^2 \geq 13Rr + r^2$$

which follows from Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$ and Euler's inequality $R \geq 2r$
The equality holds if and only if the triangle is equilateral.
The second inequality, regarding **Lemma 2**, can be written:

$$\frac{a+b+c}{5} \leq \sum (b+c-a)\cdot\frac{h_a - 2r}{h_a + 2r} \Leftrightarrow \frac{2p}{5} \leq \frac{2p(4p^2 - 3r^2 - 40Rr)}{4p^2 + r^2 + 8Rr} \Leftrightarrow$$

$$\Leftrightarrow 5(4p^2 - 3r^2 - 40Rr) \geq 4p^2 + r^2 + 8Rr \Leftrightarrow p^2 \geq 13Rr + r^2$$

which follows from Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$ and Euler's inequality $R \geq 2r$.
The equality holds if and only if the triangle is equilateral.

Marin Chirciu

Fifth solution. Let F and s be, respectively, area and semiperimeter of the triangle.
Then $h_x = \frac{2F}{x}, x \in \{a, b, c\}$ and $r = \frac{F}{s}$ and original inequality becomes

$$\sum_{cyc} \frac{a\left(\frac{2F}{a} - \frac{2F}{s}\right)}{\frac{2F}{a} + \frac{2F}{s}} \leq \frac{a+b+c}{5} \leq \sum_{cyc} \frac{2(s-a)\left(\frac{2F}{a} - \frac{2F}{s}\right)}{\frac{2F}{a} + \frac{2F}{s}} \Longleftrightarrow$$

$$\Longleftrightarrow \sum_{cyc} \frac{a(s-a)}{s+a} \leq \frac{2s}{5} \leq \sum_{cyc} \frac{2(s-a)^2}{s+a}. \tag{1}$$

1. Left hand side of (1):

$$\sum_{cyc} \frac{a(s-a)}{s+a} \leq \frac{2s}{5} \Longleftrightarrow \sum_{cyc}\left(a + \frac{a(s-a)}{s+a}\right) \leq \frac{12s}{5} \Longleftrightarrow$$

$$\Longleftrightarrow \sum_{cyc} \frac{2as}{s+a} \leq \frac{12s}{5} \Longleftrightarrow \sum_{cyc} \frac{a}{s+a} \leq \frac{6}{5} \Longleftrightarrow \sum_{cyc}\left(1 - \frac{a}{s+a}\right) \geq 3 - \frac{6}{5} \Longleftrightarrow$$

$$\Longleftrightarrow 5s \sum_{cyc} \frac{1}{s+a} \geq 9 \Longleftrightarrow \sum_{cyc}(s+a) \cdot \sum_{cyc} \frac{1}{s+a} \geq 9$$

(by Cauchy Inequality).
2. Right hand side of (1):

$$\frac{2s}{5} \leq \sum_{cyc} \frac{2(s-a)^2}{s+a} \iff \frac{s}{5} \leq \sum_{cyc} \frac{(s-a)^2}{s+a}$$

and by Cauchy Inequality

$$\sum_{cyc}(s+a)\cdot\sum_{cyc}\frac{(s-a)^2}{s+a} \geq \left(\sum_{cyc}(s-a)\right)^2 \iff 5s\cdot\sum_{cyc}\frac{(s-a)^2}{s+a} \geq s^2 \iff \frac{s}{5}.$$

Arkady Alt

Sixth solution. L.H.S. Letting Δ be area of the triangle, we know that

$$ah_a = 2\Delta, \quad \Delta = rs$$

so the inequality is

$$\sum \frac{a(\frac{2rs}{a}-2r)}{\frac{2rs}{a}+2r} \leq \frac{a+b+c}{5} \iff \sum \frac{a(s-a)}{s+a} \leq \frac{2s}{5}$$

The l.h.s. is concave since, after fixing the value of s, we have

$$\left(\sum \frac{a(s-a)}{s+a}\right)'' = -\frac{4s^2}{(s+x)^3}$$

so

$$\sum \frac{a(s-a)}{s+a} \leq \sum \frac{\frac{2}{s}3(s-\frac{2s}{3})}{s+\frac{2s}{3}} = \frac{2s}{5}$$

R.H.S. The inequality reads as

$$\sum \frac{(b+c-a)\left(\frac{2rs}{a}-2r\right)}{\frac{2rs}{a}+2r} = \sum \frac{(a+b+c-2a)(s-a)}{s+a} \geq \frac{a+b+c}{5}$$

or

$$\sum \frac{(s-a)^2}{s+a} \geq \frac{2s}{5}$$

Again we fix the value of s and observe

$$\left(\frac{(s-a)^2}{s+a}\right)'' = \frac{8s^2}{(s+x)^3} \geq 0$$

so